Let r' be the radius of sphere whose volume is equal to the volume of the solid sector. Then $\frac{4}{3}\pi r'^3 = \frac{1000}{3}\pi$.

$$\therefore\ r' = \sqrt[3]{250} = 5\sqrt[3]{2}.$$

Also solved by A. M. Harding, Paul Capron, Walter C. Eells, Elmer Schuyler, H. C. Feemster, Horace Olson, C. E. Githens, Geo. W. Hartwell, and Clifford N. Mills.

CALCULUS.

354. Proposed by C. N. SCHMALL, New York City.

Prove

$$\Gamma(1 + a)\Gamma(1 - a) = \frac{\pi a}{\sin \pi a}.$$

Solution by A. G. Caris, Defiance, O.

(The outline for this proof is taken from notes of a course in *Definite Integrals* given by Professor G. A. Bliss at the University of Chicago.)

The following preliminary theorems are used in the proof.

$$\sin z = z\prod_{\nu=1}^{\infty}\left(1 - \frac{z^2}{\nu^2\pi^2}\right). \tag{1}$$

$$\cos z = \prod_{\nu=1}^{\infty}\left(1 - \frac{4z^2}{(2\nu - 1)^2\pi^2}\right). \tag{2}$$

$$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots. \tag{3}$$

$$\int_0^{\infty} \frac{x^{a-1}}{1 + x}\,dx = \frac{\pi}{\sin \pi a}. \tag{4}$$

$$\Gamma(1 + a) = a\Gamma(a). \tag{5}$$

$$y^{-a}\Gamma(a) = \int_0^{\infty} x^{a-1}e^{-yx}dx. \tag{6}$$

Proof of Theorem (1).

$$\sin z = 2\sin\frac{z}{2}\cos\frac{z}{2} = 2\sin\frac{z}{2}\sin\frac{z + \pi}{2}.$$

By repeated applications of this formula we obtain

$$\sin z = 2^{2^n-1}\sin\frac{z}{2^n}\cdot\sin\frac{z + \pi}{2^n}\cdot\sin\frac{z + 2\pi}{2^n}\cdots\sin\frac{z + (2^n - 1)\pi}{2^n}.$$

This may be written more briefly by substituting $2^n = p$.

$$\sin z = 2^{p-1}\prod_{\nu=0}^{p-1}\sin\frac{z + \nu\pi}{p}.$$

Combine these factors in pairs, taking together such factors that the sum of the coefficients of π shall be p. The general form of these products will be

$$\sin\frac{z+\nu\pi}{p}\sin\frac{z+(p-\nu)\pi}{p}.$$

By trigonometric transformation, this becomes

$$\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}.$$

There will be $\left(\frac{p}{2}-1\right)$ of these factors, the factor $\sin\frac{z}{p}$, and the factor $\sin\frac{z+\frac{p}{2}\pi}{p}$ which equals $\cos\frac{z}{p}$.

Whence,
$$\sin z = 2^{p-1}\sin\frac{z}{p}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}\right). \tag{A}$$

Dividing both sides of this equation by z,

$$\frac{\sin z}{z}=\frac{2^{p-1}}{p}\cdot\frac{\sin\frac{z}{p}}{\frac{z}{p}}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(\sin^2\frac{\nu\pi}{p}-\sin^2\frac{z}{p}\right).$$

Passing to the limit of both sides as $z \doteq 0$, we have

$$1=\frac{2^{p-1}}{p}\prod_{\nu=1}^{(p/2)-1}\sin^2\frac{\nu\pi}{p}.$$

Dividing both members of (A) by this result, we have

$$\sin z = p\sin\frac{z}{p}\cos\frac{z}{p}\prod_{\nu=1}^{(p/2)-1}\left(1-\frac{\sin^2\frac{z}{p}}{\sin^2\frac{\nu\pi}{p}}\right).$$

Taking the limit of this expression as $p \doteq \infty$, we have

$$\sin z = z\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2\pi^2}\right).$$

Proof of Theorem (2).

$$\sin z = 2\sin\frac{z}{2}\cos\frac{z}{2}=z\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{\nu^2\cdot\pi^2}\right).$$

From (1),

$$\sin\frac{z}{2}=\frac{z}{2}\prod_{\nu=1}^{\infty}\left(1-\frac{z^2}{(2\nu)^2\pi^2)}\right).$$

Dividing the first of these two equations by the second, we have

$$2 \cos \frac{z}{2} = 2 \prod_{\nu=1}^{\infty} \left(1 - \frac{z^2}{(2\nu - 1)^2 \pi^2} \right).$$

From this we have at once

$$\cos z = \prod_{\nu=1}^{\infty} \left(1 - \frac{4z^2}{(2\nu - 1)^2 \pi^2} \right).$$

PROOF OF THEOREM (3).

From (1) and (2) we have

$$\sin \pi a = \pi a \prod_{\nu=1}^{\infty} \left(1 - \frac{a^2}{\nu^2} \right), \qquad \cos \pi a = \prod_{\nu=1}^{\infty} \left(1 - \frac{4a^2}{(2\nu - 1)^2} \right).$$

At once we have

$$\log \sin \pi a = \log \pi + \log a + \log \left(1 - \frac{a^2}{1^2} \right) + \log \left(1 - \frac{a^2}{2^2} \right) + \cdots$$

and

$$\log \cos \pi a = \log \left(1 - \frac{4a^2}{1^2} \right) + \log \left(1 - \frac{4a^2}{3^2} \right) + \log \left(1 - \frac{4a^2}{5^2} \right) + \cdots.$$

By differentiation of these two series, we have

$$\pi \cot \pi a = \frac{1}{a} - \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} - \frac{2a}{3^2 - a^2} - \cdots$$

$$\pi \tan \pi a = \frac{8a}{1^2 - 4a^2} + \frac{8a}{3^2 - 4a^2} + \frac{8a}{5^2 - 4a^2} + \cdots.$$

Substituting $a/2$ for a, we have

$$\pi \tan \frac{\pi a}{2} = \frac{4a}{1^2 - a^2} + \frac{4a}{3^2 - a^2} + \frac{4a}{5^2 - a^2} + \cdots.$$

$$\pi \tan \frac{\pi a}{2} + \pi \cot \pi a = \pi \left[\frac{\sin \frac{\pi a}{2}}{\cos \frac{\pi a}{2}} + \frac{\cos^2 \frac{\pi a}{2} - \sin^2 \frac{\pi a}{2}}{2 \sin \frac{\pi a}{2} \cos \frac{\pi a}{2}} \right] = \frac{\pi}{\sin \pi a}$$

Adding the corresponding series, we have

$$\frac{\pi}{\sin \pi a} = \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

PROOF OF THEOREM (4).

$$\int_0^\alpha \frac{x^{a-1}}{1 + x} dx = \int_0^1 \frac{x^{a-1}}{1 + x} dx + \int_1^\alpha \frac{x^{a-1}}{1 + x} dx.$$

Transforming the second integral by the substitution, $x = \frac{1}{y}$,

$$\int_1^a \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \frac{y^{-a}}{1+y}\,dy,$$

$$\int_0^1 \frac{x^{a-1}}{1+x}\,dx = \int_0^1 \{x^{a-1} - x^a + x^{a+1} - x^{a+2} + \cdots\}\,dx,$$

$$\int_0^1 \frac{y^{-a}}{1+y}\,dy = \int_0^1 \{y^{-a} - y^{-a+1} + y^{-a+2} - \cdots\}\,dy.$$

Integrating these two series term by term, we have,

$$\int_0^a \frac{x^{a-1}}{1+x}\,dx = \left\{\frac{1}{a} - \frac{1}{a+1} + \frac{1}{a+2} - \cdots\right\} + \left\{\frac{1}{1-a} - \frac{1}{2-a} + \frac{1}{3-a} - \cdots\right\}$$

$$= \frac{1}{a} + \frac{2a}{1^2 - a^2} - \frac{2a}{2^2 - a^2} + \frac{2a}{3^2 - a^2} - \cdots.$$

But this, by (3), is $\frac{\pi}{\sin \pi a}$.

$$\therefore \quad \int_0^a \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin \pi a}.$$

Proofs of theorems (5) and (6) may be found in almost any elementary discussion of the properties of $\Gamma(a)$, and need not be given here.

We are now to prove

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

By (5), $\Gamma(1+a) = a\Gamma(a)$.

Our problem is thus reduced to the proof of

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}.$$

By (6),
$$\frac{\Gamma(a)}{y^a} = \int_0^a x^{a-1}e^{-yx}dx.$$

Multiplying both sides by e^{-y}, we have

$$\frac{e^{-y}\Gamma(a)}{y^a} = \int_0^a x^{a-1}e^{-y(x+1)}dx.$$

Taking the integral of both sides with respect to y from 0 to a, we have

$$\Gamma(a)\int_0^a \frac{e^{-y}}{y^a}\,dy = \int_0^a\int_0^a x^{a-1}e^{-y(x+1)}dxdy = \int_0^a x^{a-1}\int_0^a e^{-y(x+1)}dydx$$

$$= \int_0^a \frac{x^{a-1}}{1+x}\,dx.$$

By (4),

$$\int_0^a \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin \pi a}.$$

$$\int_0^a \frac{e^{-y}}{y^a}dy = \int_0^a y^{-a}e^{-y}dy = \int_0^a y^{(1-a)-1}e^{-y}dy = \Gamma(1-a).$$

$$\therefore \quad \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a},$$

and

$$\Gamma(1+a)\Gamma(1-a) = \frac{\pi a}{\sin \pi a}.$$

Also solved by T. M. Blakslee, C. N. Schmall, A. M. Harding, A. L. McCarty, and J. W. Clawson.

355. Proposed by C. N. SCHMALL, New York City.

Given the curve of the nth degree,

$$y^n - (a+bx)y^{n-1} + (c+dx+ex^2)y^{n-2} + \cdots = 0,$$

show that if each ordinate is divided by the corresponding subtangent, the sum of all the resulting ratios will be a constant.

Solution by J. W. Clawson, Collegeville, Pa.

The question should read: "show that if for a given abscissa each ordinate . . . ."

The sum of all the ordinates corresponding to a given abscissa x_1 is equal to minus the coefficient of y^{n-1}, viz., $+ (a + bx_1)$.

Hence, the sum of the derivatives of the several ordinates will be $+ b$.

But each subtangent is the ordinate divided by the slope of the curve at the top of the ordinate. Hence any ordinate divided by its corresponding subtangent is equal to the slope of the curve at the top of that ordinate. We have just shown that the sum of these slopes for all the ordinates corresponding to a given abscissa is a constant. This proves the problem, as amended. See Edwards' *Differential Calculus*, page 151.

Also solved by the Proposer.

MECHANICS.

274. Proposed by G. B. M. ZERR.

A sphere moves on the concave side of a rough cylindrical surface of which the transverse section perpendicular to the generating lines is a hypocycloid. If $s = l \sin n\theta$ be the intrinsic equation of the hypocycloid, then $l = (a-b)4b/a$, $n = a/(a-2b)$, where a = radius of fixed circle, b = radius of rolling circle.

Remark by A. H. Wilson, Haverford College.

The statement of this problem is incomplete. As it stands it is not a problem at all.

275. Proposed by W. J. GREENSTREET, Editor of the Mathematical Gazette, England.

If a particle be attracted towards the angular points of a regular hexagon by forces equal to r^{-h}, at distance r, find the condition for stability of equilibrium.